HEAT TRANSFER *in* COMPOSITE MATERIALS

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DES*tech* Publications, Inc.

Heat Transfer in Composite Materials

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Lancaster, Pennsylvania 17602 U.S.A.

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Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

Main entry under title:
Heat Transfer in Composite Materials

A DEStech Publications book
Bibliography: p.
Includes index p. 207

Library of Congress Control Number: 2017956643
ISBN No. 978-1-60595-459-2

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Preface

THE use of composite materials in industry began in the 1940s with GFRP (glass fiber reinforced plastics). Since then, the research and development of composite materials have gone through significant breakthroughs turning the subject of composite materials into a matured discipline in applied science today. The most compelling reason for industrial use of composite materials is to take advantage of their anisotropic nature which enables such material properties as strength, stiffness and fracture toughness to be tailored to specific directions.

Composites often fail when they are subject to severe environments such as high temperatures or cryogenic temperatures even though there is no external load applied to the composites. Typically, failure in composites is caused by thermal stresses which are generated at the interface between the reinforcing fibers and the surrounding matrix due to the mismatch of the thermal expansion coefficients of the two materials. The mismatch of thermal expansion coefficients is not the only cause for thermal stresses. Thermal stresses are also generated by the mismatch of the thermal conductivities and the elastic stiffness if the temperature distribution in the composite is non-uniform.

In order to understand how the thermal stress is generated and distributed in the composites, it is necessary to know the temperature distribution in the composite first. It is also important to know the effective thermal conductivity of the composite when it is viewed as an equivalent homogeneous medium that exhibits the same response as the composite. While the mechanical behavior of composite materials has been investigated for decades, research on the thermal properties of composites is somewhat less explored, which motivated the authors to write this book.

One of the authors (SN) has been working on developing analytical methods for the mechanical properties of composites using micromechanics approaches in which microstructures such as inclusions, defects and dislocations are taken into account that affect the properties of heterogeneous materials. The other author (AHS) has been working on a wide range of heat transfer research and developed semi-analytical methods to accurately predict the temperature distribution in anisotropic media for both steady-state and transient state. Therefore it was logical that both of us decided to write a book on analytical methods for heat transfer in composites where the availability of books on this topic was scarce in the composite research community. Steady-state heat transfer in composites composed of inclusions (fibers) and a matrix phase of both finite and infinite size as well as the thermal stress resulting from the temperature distribution were handled by SN and both steady-state and transient-state heat transfer for muli-layer composites was handled by AHS.

The purpose of this book is to introduce analytical methods that can be used to obtain temperature distributions in general heterogeneous materials. Although many heat transfer problems in composites are routinely solved by numerical methods such as the finite element method or the finite difference method, analytical solutions are always preferred to numerical solutions. The readers are expected to have basic mathematical background at the undergraduate level of vector calculus, linear algebra and differential equations. As many of the formulas are lengthy and tedious, it is helpful if the readers have access to computer algebra software such as Mathematica and Maple that automates symbolic derivations of many mathematical formulas. For steady-state heat conduction, the governing equation for heat conduction is similar to the equations for permeability, electrical conductivity, diffusivity, dielectric constant, and magnetic permeability. However, the transient behavior of heat conduction significantly differs from the properties above which is handled in Chapter 4.

The book consists of seven Chapters. Chapter 1 is a general introduction and review of the heat conduction equations. Exemplar problems are solved that represent the basics of analytical methods employed in the subsequent Chapters. In Chapter 2, steady-state heat conduction in composites reinforced by fibers/particulates is solved from the micromechanics viewpoint. The temperature distribution in a medium that contains a spheroidal inclusion is sought and it is expressed analytically for an unbounded medium and semi-analytically for a bounded medium. A spheroidal inclusion can cover a wide range of fiber shapes from a flat-flake to a sphere to a long fiber. Chapter 3 discusses the analytical solutions for steady-state heat conduction in laminated multilayer composites and in heterogeneous materials. In Chapter 4, different analytical solutions for transient heat conduction in multilayer and laminated composites are emphasized. Chapter 5 presents an overview of rapid energy transport in heterogeneous composites under a local thermal non-equilibrium condition. Chapter 6 discusses the effective thermal conductivity of unbounded composite materials by introducing critical theoretical models including the upper and lower bounds of Hashin and Shtrikman, the Maxwell-Garnett effective medium theory, the Mori-Tanaka model and the self-consistent approximation. Chapter 7 discusses thermal stresses caused by a mismatch of the thermal expansion coefficients at the interface of the matrix and fibers due to a non-uniform temperature distribution in the composites. Although thermal stress analysis is not a subject within heat transfer, it is an important topic as the composites often fail due to non-uniform temperature distributions. The thermal stress analysis is carried out based on the results from the preceding Chapters. References are given at the end of each Chapter.

We wish to thank Dr. Joseph Eckenrode and Mr. Stephen Spangler at DEStech Publications for their encouragement and support throughout the period of this book project.

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CHAPTER 1

Basic Equations for Heat Transfer

1.1. FOURIER'S LAW

In this Chapter, the fundamental equations of heat transfer are derived and well-known exemplar problems are solved. As this book is not meant to be a general textbook on heat transfer, the readers are referred to one of many outstanding textbooks (e.g. [1]) and only a minimum amount of equations to be used in the subsequent Chapters are presented. However, the formulations are intended to be applicable to anisotropic and heterogeneous materials later that define the composite materials. Although a preferred way of describing the mechanical and physical properties of anisotropic materials is to use the tensor (index) notation, it is not employed in this book except for Chapter 7 to avoid unnecessary complexity. More discussion on heat conduction in composites using index notation is found in [2].

The heat conduction equation is derived from Fourier's law. Fourier's law is an empirical relationship between the heat flow and temperature gradient. Fourier's law states that the heat flows when there is a temperature difference between two points (Figure 1.1). The amount of the flow is proportional to the temperature difference and the direction of the flow is also related to the direction of the temperature difference, which can be expressed as

$$\mathbf{q} \propto \nabla T \tag{1.1}$$

where $\mathbf{q}$ is the heat flux (a vector) whose dimension is w/m^2, T is the

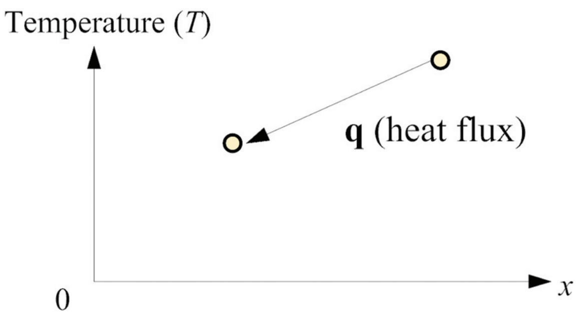

FIGURE 1.1. *Fourier's law stating that heat flows from a point of higher temperature to a point of lower temperature.*

temperature (a scalar) and ∇T (a vector) is the gradient of T. Mathematically, Equation (1.1) can be expressed as

$$\mathbf{q} = -K\nabla T \tag{1.2}$$

where K is a 3 × 3 matrix which is the proportionality factor between the heat flux, **q**, and the temperature gradient, ∇T, and is called thermal conductivity with the dimension, w/(m·k). Equation (1.2) is explicitly expressed as

$$\begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = -\begin{pmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{pmatrix} \begin{pmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{pmatrix} \tag{1.3}$$

The minus sign in Equation (1.2) comes from the fact that heat flows from a point of higher temperature to a point of lower temperature. It should be noted that according to Onsager's principle [3], K is symmetrical as

$$k_{xy} = k_{yx}, \quad k_{yz} = k_{zy}, \quad k_{zx} = k_{xz} \tag{1.4}$$

Hence, the number of independent components for K is 6 in 3-D (4 in 2-D).

The general definition of composite materials is any medium which is heterogeneous. However, in engineering, composite materials are re-

CHAPTER 2

Heat Conduction in Matrix-Inclusion/ Fiber Composites

2.1. INTRODUCTION

THE heat conduction equation [Equation (1.12)] along with an appropriate boundary condition constitutes a boundary value problem. One of the challenges of solving Equation (1.12) is when the thermal conductivity, k, is a function of positions, which defines composite materials. Typical composite materials are defined by their mechanical and physical properties such as the thermal conductivities, $k(x)$, that are piecewise constant across different phases. As many analytical methods for heat transfer problems that have been developed for conventional homogeneous materials cannot be readily used for composite materials, it is necessary to develop tailored techniques for the composites.

In Section 2.2, analytical methods to obtain the temperature distribution in an infinitely extended medium that contains an inclusion (a second phase) are presented. The shape of the inclusion can be spherical, cylindrical and spheroidal (an ellipsoid with two equal semi-diameters). The temperature distribution in a medium containing a spherical (3-D) or circular (2-D) inclusion is derived using two different approaches, first by solving the partial differential equations directly and second by using the complex variable theory. A three-phase problem where a single inclusion is surrounded by another inclusion of the same spherical shape is also solved using the aforementioned two approaches. For a medium that contains a spheroidal inclusion, a different approach is employed using the Green's function. A spheroidal inclusion can cover a wide range of shapes from a flat flake to a sphere to a long fiber. As will be seen in Chapter 6, the effective thermal conductivity of the composite that contains spheroidal inclusions can be obtained if the temperature field inside the inclusion is known.

In Section 2.3, the 2-D steady-state heat conduction problem is solved in which a circular inclusion is embedded in a finite sized medium. It is noted that if the medium that contains an inclusion is infinitely extended, analytical solutions are available as shown in Section 2.2. However, if the medium is of finite size and the geometry is not axi-symmetrical, there is no analytical solution available in general. Although numerical methods such as the finite element method have been routinely used, analytical or semi-analytical methods are always preferred. In Section 2.3, a semi-analytical method is presented to derive the temperature field in series form using the Galerkin method. The permissible functions that are used in the Galerkin method are analytically derived which satisfy both the continuity condition of the temperature and the heat flux at the interface. It is noted that test functions used in the finite element method do not satisfy the continuity condition of heat flux across the interface. Unlike numerical methods, this semi-analytical method retains all the material and geometrical parameters in the formulation, thus, suitable for parametric study.

2.2. SPHERICAL/CYLINDRICAL INCLUSION PROBLEMS IN AN UNBOUNDED MEDIUM

In this section, the steady-state heat conduction equation is solved when a spherical (3-D) inclusion or a cylindrical inclusion (2-D) is embedded in an infinitely-extended matrix phase subject to a constant heat flux at the far field. As will be discussed in Chapter 6, the solution for a single inclusion can be directly used to obtain the effective thermal conductivity for a composite that contains multiple inclusions.

2.2.1. Spherical Inclusion (3-D)

A 3-D unbounded isotropic medium is considered that contains a spherical inclusion with the radius, a, at the center subject to a uniform heat flow, $\mathbf{q}^{\infty}$, at infinity as shown in Figure 2.1. The heat flux approaches a constant vector, $\mathbf{q}^{\infty}$, as $x, y, z \rightarrow \pm\infty$ expressed as

$$\left(k^m \frac{\partial T}{\partial x}, k^m \frac{\partial T}{\partial y}, k^m \frac{\partial T}{\partial z}\right) \to \left(q_x^\infty, q_y^\infty, q_z^\infty\right) \quad x, y, z \to \pm\infty \tag{2.1}$$

where k^m is the thermal conductivity for the matrix.

As the only source for the temperature distribution is the uniform heat flux, $\mathbf{q}^\infty$, at infinity, the temperature field throughout the material must be proportional to $\mathbf{q}^\infty$. Therefore, the only possible expression for the temperature field is to assume that

$$T = f(r)\mathbf{q}^\infty \cdot \mathbf{x} \tag{2.2}$$

where $f(r)$ is an unknown function yet to be determined but a function of the distance, $r = \sqrt{\mathbf{x} \cdot \mathbf{x}}$, alone and $\mathbf{q}^\infty \cdot \mathbf{x}$ is the dot product between the heat flux, $\mathbf{q}^\infty$, and $\mathbf{x}$, $q_x x + q_y y + q_z z$. As no heat source is assumed, the temperature field, T, in both the inclusion and the matrix is subjected to the Laplace equation as

$$\Delta T = 0 \tag{2.3}$$

where Δ is the Laplacian defined as

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.4}$$

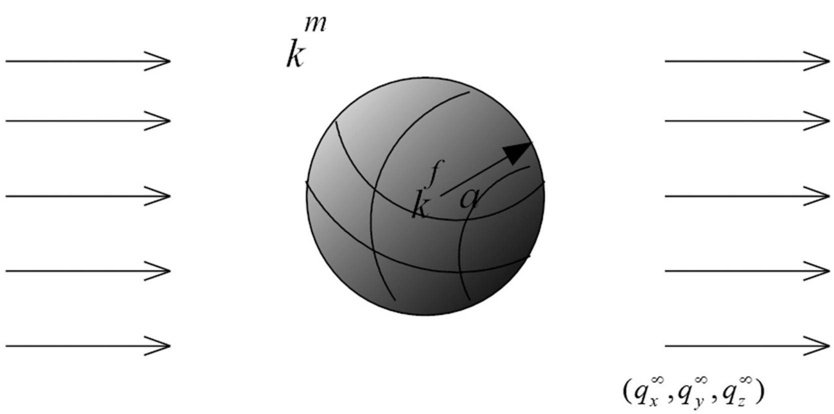

FIGURE 2.1. *A spherical inclusion in an unbounded matrix phase subject to uniform heat flux at infinity.*

Steady State Heat Conduction in Multi-Layer Composite Materials

3.1. INTRODUCTION

HEAT conduction in multi-layer composite materials appears in various applications, such as cooling systems to protect electronic devices. In this Chapter, the analytical solutions of temperature fields in multi-dimensional regions are emphasized. This includes temperature solutions in multi-layered orthotropic bodies with rectangular profiles. The steady state solutions are presented in this Chapter and they are followed by the transient solutions in Chapter 4.

Earlier works on analytical solutions of heat conduction problems in composite materials are available in the literature. A one-dimensional orthogonal expansion for a composite material is reported in [1]. Later, a multi-dimensional orthogonal expansion for composite materials is reported in [2]. Furthermore, a generalized Sturm-Liouville procedure to solve transient heat conduction problems for composites as well anisotropic media was developed and presented in [3]. Others [4–9] examined the transient temperature solution in a two- and three-dimensional isotropic-composite slab. As reported in [9], the computation of temperature in multi-dimensional and multi-layer bodies has special features to be discussed in the next Chapter.

In this Chapter, consideration is given to the determination of the steady-state temperature field in multi-dimensional orthotropic bodies. This methodology is extended and applied to multi-layer bodies. The computations include the contribution of contact resistance between adjacent layers. This Chapter includes the selection method for eigenfunctions to be followed by the computation of the eigenvalues.

Figure 3.1 shows a schematic of multi-layer composites. In the absence of a volumetric heat source, it is acceptable to use superposition of the temperature solutions by dividing this solution technique into two parts: The first part considers non-homogeneous boundary conditions at $y = 0$ and/or $y = b$ while all other boundary conditions are homogeneous. In the second part, the boundary conditions at $y = 0$ and/or $y = b$ are homogeneous while one of the boundaries in x- or z-direction can have a non-homogeneous condition. Accordingly, the mathematical formulations presented in this Chapter are for three separate solutions. The first one describes the solution for a problem with non-homogeneous boundary conditions along $y = 0$ or $y = b$ surface. This leads to a standard solution technique and a brief presentation is provided. The emphasis of the second solution is directed toward the case when one or more non-homogeneous boundary conditions exist along the surfaces $x = 0$, $x = a$, $z = 0$, or $z = c$. The third solution describes the contribution of a volumetric heat source and this solution technique uses the eigenvalue determinations presented in the previous two cases. The described mathematical procedure is applied to a set of selected examples.

The determination of thermophysical property is widely available in the literature. A classical thermodynamic relation can produce the specific heats with reasonable accuracy. The thermal conductivity values of composites are often determined experimentally as a function of temperature. Experimental and theoretical studies related to the composite

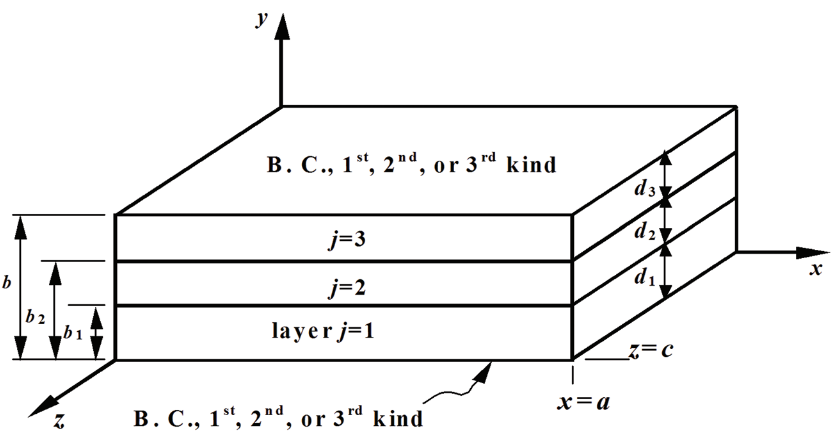

FIGURE 3.1. Schematic of a multi-layer body.

CHAPTER 4

Transient Heat Conduction in Multi-Layer Composite Materials

4.1. INTRODUCTION

TRANSIENT heat conduction in multi-layer materials appears in various engineering cooling systems for different applications. Earlier works on analytical solutions of heat conduction problems include a one-dimensional orthogonal expansion for a composite medium reported in [1]. Later, a generalized Sturm-Liouville procedure to solve transient heat conduction problems for composite and anisotropic domains was developed and presented in [2]. Others [3–7] examined the transient temperature solution in the two- and three-dimensional isotropic-composite slabs. A presentation of transient conduction in multi-layer bodies is in [8] and a study of steady-state conduction in layered bodies is reported in [9]. As reported in [8], the computation of temperature in multi-dimensional and multi-layer orthotropic bodies has special features. It is noted that, for special cases, the eigenvalues may become imaginary and that would produce eigenfunctions with imaginary arguments. Also, it is reported that the spacing between successive eigenvalues may change between zero and a maximum value, in some applications; therefore, it becomes necessary to identify the location of each eigenvalue. Once an approximate location of an eigenvalue is known, there are procedures to determine the eigenvalues, as described in [10].

This Chapter includes temperature solutions in multi-layer orthotropic bodies with rectangular profiles. Consideration is given to a body with layers of different materials one above the other in the y-direction, as shown in Figure 4.1. The selected thermal conductivities for fiber

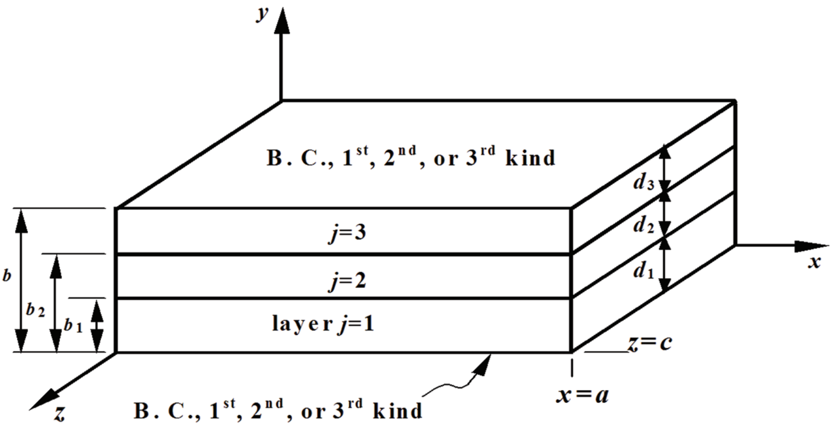

FIGURE 4.1. Schematic of a multi-layer body.

reinforced composites have the effective values and they can have directional dependence while the selected specific heat and density have the mean values. The mathematical solutions of temperature fields in multi-dimensional regions are emphasized. Furthermore, the mathematical solutions include the contribution of thermal contact resistance at the interface between the layers and it vanishes in the presence of perfect thermal contact.

As stated earlier, the computation of temperature in multi-dimensional, multi-layer bodies exhibits features not commonly observed in the temperature solutions for homogeneous bodies. Care should be exercised when computing the eigenvalues; especially when the spacing between successive eigenvalues can approach to a zero value. It practice, it is possible to identify the domain within which there is a single eigenvalue. After that, one can use a root finding technique to accurately determine the eigenvalues; e.g., the hybrid root finding technique in [10]. Additionally, this Chapter includes the Green's function solution. Further details related to these solutions are in [11,12].

4.2. MATHEMATICAL RELATIONS

The mathematical formulation for transient heat conduction in three-dimensional multi-layer bodies is the subject of this Chapter. As before, consideration is given to a multi-layer body as depicted in Figure 4.1. Analytical formulations for temperature solutions in multi-layer bodies are presented and later consideration is given to three-dimensional two-layer bodies. In each layer, the thermo-physical properties can be viewed to have their effective values; otherwise, a two-step solution is needed. Although the multi-layer bodies can have isotropic layers, the diffusion equation for an orthotropic layer j is

$$k_{x,j}\frac{\partial^2 T_j}{\partial x^2}+k_{y,j}\frac{\partial^2 T_j}{\partial y^2}+k_{z,j}\frac{\partial^2 T_j}{\partial z^2}=\rho_j c_{p,j}\frac{\partial T_j}{\partial t} \tag{4.1}$$

when

$$b_{j-1}<y<b_j$$

For homogeneous boundary conditions, one can propose a solution of the form

$$T_j(x,y,z,t)=X_j(x)Y_j(y)Z_j(z)\Gamma_j(t) \quad \text{in Layer } j \tag{4.2}$$

satisfying the following conditions:

$$X_j''/X_j=-\beta^2 \quad \text{for} \quad j=1,2,\cdots,N \tag{4.3a}$$

$$Z_j''/Z_j=-v^2 \quad \text{for} \quad j=1,2,\cdots,N \tag{4.3b}$$

$$\Gamma_j''/\Gamma_j=-\lambda^2 \quad \text{for} \quad j=1,2,\cdots,N \tag{4.3c}$$

In order to satisfy the continuity conditions, the parameters β, v, and λ must remain the same for all layers, independent of j. Therefore, each parameters β and v must depend on the specific type of homogeneous boundary conditions for all layers in the x-direction and the z-direction, respectively. Then, the differential equations for Y_j is obtainable after substitution for T_j from Equation (4.2) into Equation (4.1) with Y_j, Z_j, and Γ_j as given by Equations (4.3a-c) to yield

$$-k_{x,j}\beta^2+k_{y,j}Y_j''/Y_j-k_{z,j}v^2=\rho_j c_{p,j}(-\lambda^2) \tag{4.4}$$

that becomes

$$\frac{Y_j''}{Y_j}=-\frac{\lambda^2}{\alpha_{y,j}}+\frac{k_{x,j}}{k_{y,j}}\beta^2+\frac{k_{z,j}}{k_{y,j}}v^2=-\gamma_j^2 \tag{4.5}$$

where $\sin[2\beta_m(r_1 - r_0)] = 2\sin[\beta_m(r_1 - r_0)]\cos[2\beta_m(r_1 - r_0)]$ vanishes for the boundary conditions of the first kind at $r = r_1$, but not for the boundary conditions of the second kind at $r = r_1$.

For the boundary conditions of the second kind at $r = r_1$, the eigenvalues are obtainable by setting

$$R'(r_1) = \frac{d}{dr}\left[\frac{1}{\beta r}\sin[\beta(r - r_0)]\right]_{r=r_1} = 0 \tag{4.86a}$$

which yields

$$\frac{d}{dr}\left[\frac{1}{r}\sin[\beta(r - r_0)]\right]_{r=r_1} = \frac{\beta}{r}\cos[\beta(r_1 - r_0)] - \frac{1}{r_1}\sin[\beta(r - r_0)] = 0 \tag{4.86b}$$

and then

$$\beta(r_1 - r_0)\frac{\cos[\beta(r_1 - r_0)]}{\sin[\beta(r_1 - r_0)]} = \frac{1}{r_1}(r_1 - r_0) \tag{4.86c}$$

or

$$\beta(r_1 - r_0)\cot[\beta(r_1 - r_0)] = 1 - \frac{r_0}{r_1} \tag{4.86d}$$

Note that $\cot^2 x = (1 - \sin^2 x)/\sin^2 x = \cos^2 x/(1 - \cos^2 x)$ and this equation produces $\sin^2 x = 1/(1 + \cot^2 x)$ and $\cot^2 x = \cot^2 x/(1 - \cot^2 x)$. Then, using these basic relations to $\sin^2 x\cos^2 x = \cot^2 x/(1 + \cot^2 x)$ get that leads to the relation

$$\sin[2\beta_m(r_1 - r_0)] = \frac{r_1 - r_0}{2\beta_m^2} + \frac{\cot[\beta_m(r_1 - r_0)]}{2\beta_m^3[1 + \cot[\beta_m(r_1 - r_0)]^2]} \tag{4.87a}$$

which, as given by Equation (4.32d), yields

$$\cot[\beta_m(r_1 - r_0)] = \frac{(1 - r_0/r_1)}{\beta_m(r_1 - r_0)} \tag{4.87b}$$

4.5.3. One-Dimensional Solutions in Multi-Layer Spheres

One-dimensional mathematical statements leading to the transformed temperature solutions for $\Psi_1 = rT_1(r, t)$ and $\Psi_2 = rT_2(r, t)$ are

$$k_1 \frac{\partial^2 \Psi_1}{\partial r^2} + g_1 r = \rho_1 c_{p,1} \frac{\partial \Psi_1}{\partial t} \quad \text{when} \quad 0 < r < r_1 \tag{4.88a}$$

in Region 1 and

$$k_2 \frac{\partial^2 \Psi_2}{\partial r^2} + g_2 r = \rho_2 c_{p,2} \frac{\partial \Psi_2}{\partial t} \quad \text{when} \quad 0 < r < r_2 \tag{4.88b}$$

in Region 2. The boundary conditions, for a two-layer spherical body, are

$$\Psi_1 = 0 \quad \text{when} \quad r = 0 \tag{4.89a}$$

$$\Psi_1(r_1, t) = \Psi_2(r_1, t) \quad \text{when} \quad r = r_1 \tag{4.89b}$$

$$k_1 \left[\frac{\partial \Psi_1}{\partial r}(r_1, t) \right] = k_2 \left[\frac{\partial \Psi_2}{\partial r}(r_1, t) \right] + (k_2 - k_1)\Psi_2 \quad \text{at} \quad r = r_1 \tag{4.89c}$$

$$\Psi_2(r_2, t) = 0 \quad \text{at} \quad r = r_2 \tag{4.89d}$$

The initial conditions, if needed, are: $T_1(r, 0) = T_w$ and $T_2(r, 0) = T_w$. Let the solutions be $\Psi_1(r, t) = R_1(r)\Gamma(t)$ and $\Psi_2(r, t) = R_2(r)\Gamma(t)$, then

$$\Gamma(t) = \exp(-\lambda^2 t) \tag{4.90}$$

$$R_1 = \sin(\gamma r) \quad \text{when} \quad r \le r_1 \tag{4.91a}$$

and

$$R_2 = C\cos(\eta r) + D\sin(\eta r) \quad \text{when} \quad r > r_1 \tag{4.91b}$$

with

$$C = \sin(\gamma r_1)\cos(\eta r_1) - \begin{bmatrix} (k_1/k_2)\gamma\cos(\gamma r_1) \\ +(1 - k_1/k_2)\sin(\gamma r_1)/r_1 \end{bmatrix} \sin(\eta r_1)/\eta$$

CHAPTER 6

Effective Thermal Conductivities

6.1. INTRODUCTION

One of the important topics in the analysis of composite materials is to predict their overall properties known as the "effective properties." The effective properties are the mechanical and physical properties of a homogeneous medium whose overall response is the same as those of the composite as shown in Figure 6.1, thus, making it possible to identify the composite with an equivalent homogeneous medium. Finding the effective properties is called "homogenization" and has a long history of being investigated since the 19th century (see [1] for background information) exemplified by Maxwell for his "effective medium" theory for diffusion problems [2].

The interest in obtaining the effective properties of composites including the elastic stiffness, the thermal conductivity, the thermal expansion coefficient and other thermo-mechanical and physical properties was rekindled in the 1960s as composite materials began to emerge as the next generation materials drawing attention from industries, and many books dedicated to this topic have been published [3,4]. Hashin and Shtrikman [5] derived the upper and lower bounds of the effective elastic stiffness for multi-phase quasi-isotropic materials using the variational approach and later extended the model to transversely-isotropic materials [6]. This approach was also ported to the effective thermal conductivities [7].

Theoretically, obtaining the effective properties of heterogeneous materials is called a "many-body problem," which does not allow exact

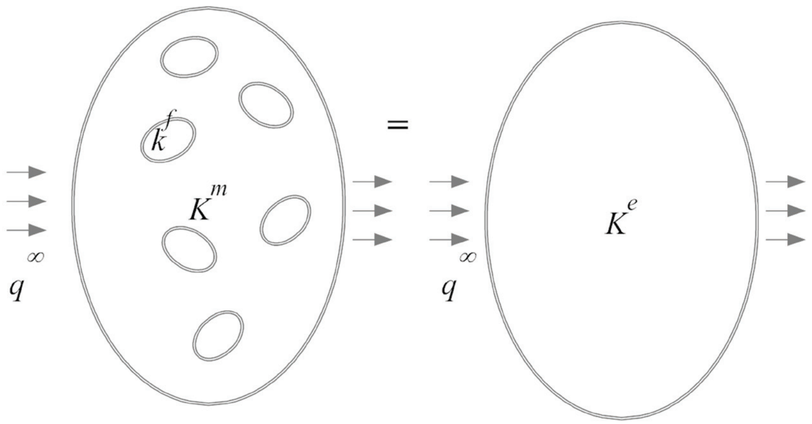

FIGURE 6.1. Definition of the effective thermal conductivities.

and closed-form solutions because it is not possible to take interaction among a number of inclusions into account. There have been numerous theories and models proposed to incorporate some interaction among multiple inclusions. In this Chapter, the effective medium theory by Maxwell [2], the Mori-Tanaka approach [8], the variational approach [5,6], and the self-consistent approximation [9–12] are presented.

The effective thermal conductivity, K^e, for an unbounded composite material is defined as

$$\bar{\mathbf{q}} = -K^e \bar{\mathbf{D}} \tag{6.1}$$

where $\mathbf{q}$ is the heat flux (a vector), $\mathbf{D}$ is the temperature gradient (a vector) and K^e is the anisotropic effective thermal conductivity (a matrix). The volume average of a function, $h(\mathbf{x})$, is denoted as $\bar{h}$ defined by

$$\bar{h} \equiv \lim_{\Omega \to \infty} \frac{1}{\Omega} \int_\Omega h(\mathbf{x}) dV \tag{6.2}$$

where Ω is the entire volume of the composite and dV is the volume element. It is known that the volume average of the temperature gradient, $\bar{\mathbf{D}}$, is equal to the temperature gradient on the boundary, $\mathbf{D}^\infty$, i.e.,

$$\bar{\mathbf{D}} = \mathbf{D}^\infty \tag{6.3}$$

Equation (6.3) can be proven using the Gauss divergence theorem as

CHAPTER 7

Thermal Stresses in Composites by Heat Flow

7.1. INTRODUCTION

THERMAL stress analysis for composite materials is important as the composite is often used in high temperature environments and thermal stresses resulting from a mismatch of thermal expansion coefficients or thermal conductivities across the interface between the fiber and matrix phases may cause catastrophic failure. One of the goals of obtaining the temperature distribution in composites is to predict the thermal stresses arising from the non-uniform temperature distribution and thus possibly preventing the composites from failure due to excessive thermal stresses.

Thermal stress problems belong to elasticity and are not the subject of this book. However, thermal stress analysis is based on prior knowledge of the temperature distribution in the composite, and many studies assume that the temperature is uniform throughout which is not the case with the composite. As it is not the mission of this book to elaborate on elasticity analysis, only a minimum amount of formulations in elasticity is presented. More detailed discussions are found in [1] and others. In this Chapter, the thermal stress is solved analytically for an infinitely extended elastic medium that contains a spherical inclusion with different material properties from the surrounding matrix phase subject to uniform heat flux at infinity based on the results in Chapter 2, in which the temperature field was derived for both the inclusion and the matrix phases under the same condition. It is clear that if a homogeneous body is placed in a uniform heat flow, there is no stress induced by the heat flow. However, if either the thermal conductivity or the thermal expansion coefficient of the inclusion is different from that of the surrounding matrix, a non-zero stress field is generated due to the presence of the heat flux. From the analysis standpoint, thermal stress analysis is equivalent to a general elasticity problem with a body force present, because the temperature effect is deemed to be a special case of a body force and hence if the elasticity equilibrium equation with a body force is solved, thermal stress problems are automatically solved using the same procedure ([2,3] for example). Other approaches include the use of complex variable theory such as [4], but they are limited to two-dimensional problems.

The solution technique employed in this Chapter is to assume that the displacement field in the medium is expressed in a form proportional to the temperature gradient at infinity since the temperature gradient (heat flux) at infinity is the only source for the induced stress [5]. Under this assumption, the displacement field for the displacement equilibrium equation is solved analytically based on the solution for the steady-state temperature distribution obtained in Chapter 2. The thermal stress field is then obtained after imposing the condition that the displacement and surface traction at the interface of the inclusion and the matrix are continuous.

7.2. REVIEW OF THERMAL STRESSES

In the absence of applied stress, a temperature change, T (a scalar), in an elastic body induces a strain field, ε (a second rank tensor). The simplest relationship between the strain, ε, and the temperature change, T, is to assume that they are linearly proportional each other with α (a second rank tensor) as the proportionality factor expressed as

$$\boldsymbol{\varepsilon} = \boldsymbol{\alpha} T \tag{7.1}$$

or equivalently in components expressed as

$$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{xy} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{xz} & \alpha_{yz} & \alpha_{zz} \end{pmatrix} T \tag{7.2}$$

The strain, $\boldsymbol{\varepsilon}$, is the infinitesimal strain whose components are related to the displacement components (u, v, w), as